Weak C*-Hopf Algebras and Multiplicative Isometries

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Abstract

We show how the data of a finite dimensional weak C^*-Hopf algebra can be encoded into a pair $(\mathcal{H}, V)$ where $\mathcal{H}$ is a finite dimensional Hilbert space and $V : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is a partial isometry satisfying, among others, the pentagon equation. In case of V being unitary we recover the Baaj-Skandalis multiplicative unitary of the discrete compact type. Relation to the pseudomultiplicative unitary approach proposed by J.-M. Vallin and M. Enock is also discussed.

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1 Introduction

The fundamental operator in Kac algebra theory [4] or the multiplicative unitary in C^*-Hopf algebras [1] is a unitary operator $V: \mathcal{H}\otimes\mathcal{H} \to \mathcal{H}\otimes\mathcal{H}$ satisfying the pentagon equation $V_{23}V_{12} = V_{12}V_{13}V_{23}$ on the three-fold tensor product of the Hilbert space $\mathcal{H}$. It encodes information about the structure of a quantum group A and its dual $\hat{A}$ in a symmetric way. If $\mathcal{H}$ is finite dimensional then a multiplicative unitary is the complete information necessary to determine a unique finite dimensional C^*-Hopf algebra [1]. In the infinite dimensional case additional assumptions are necessary: These are the regularity and irreducibility assumptions in the work of Baaj and Skandalis.

If A is a finite dimensional C^*-Hopf algebra then a multiplicative unitary on the Hilbert space of the left regular representation can be given by the formula $V(x\otimes y) = x_{(1)}\otimes x_{(2)}y$ where $x \mapsto \Delta(x) \equiv x_{(1)}\otimes x_{(2)}$ denotes the coproduct on A and $x, y \in \mathcal{H} \equiv A$. As it has been noticed in [2] if A is only a weak C^*-Hopf algebra then the V defined by the same formula still satisfies the pentagon equation but it is only a partial isometry. The purpose of the present paper is to give necessary and sufficient conditions for an operator $V: \mathcal{H}\otimes\mathcal{H} \to \mathcal{H}\otimes\mathcal{H}$ to determine a C^*-weak Hopf algebra.

C^*-weak Hopf algebras (WHA) are finite dimensional "quantum groups" with coproduct, counit, and antipode, but have no 1-dimensional representations in general. Thus the counit is not an algebra map and the antipode axioms have to be weakened accordingly. For its axioms see [2, 12] and for a detailed exposition of these quantum groups we refer to [3]. The main advantage of WHA's in describing, for instance, the symmetry of the superselection sectors in low dimensional QFT, is the flexibility of their representation theory. Given any rigid monoidal C^*-category $\mathbf{C}$ with finitely many irreducible objects one can construct a C^*-weak Hopf algebra A with representation category equivalent to $\mathbf{C}$. Roughly speaking this means that C^*-WHA's exist for arbitrary (finite) set of 6j-symbols. Since the 6j-symbols do not determine a unique C^*-WHA, one has to supply more data than just a category. These data are provided for example by a finite index depth 2 inclusion $N \subset M$ of von Neumann algebras with finite dimensional centers [9]. For II_1 factors and weak Kac algebras see [6, 7].

In a recent paper [5] M. Enock and J.-M. Vallin study the situation of a general depth 2 inclusion of von Neumann algebras with a regular operator valued weight and construct a certain isometry called a pseudo-multiplicative unitary [13]. In the finite index case it is worth to compare their construction with ours.

In Section 6 we discuss the relation of finite dimensional pseudo-multiplicative unitaries to multiplicative isometries and reveal also some connection with Ocneanu's non-Abelian cohomology [10]. It will be shown that a unital multiplicative partial isometry $V:\mathcal{H}\otimes\mathcal{H}\to\mathcal{H}\otimes\mathcal{H}$, what we introduce in Sections 2 and 3, always determines a pseudo-multiplicative unitary $U:\mathcal{H}\boxslash\mathcal{H}\to\mathcal{H}\boxbslash\mathcal{H}$. By the results of Section 3 this situation corresponds to the case when the 'right leg' and 'left leg' of V, the algebras A and $\hat{A}$, respectively, are weak bialgebras in the sense used in [3]. In Section 4 we put stronger conditions on V and assume that it satisfies a regularity condition, generalizing the one of [1]. Then we show that A and $\hat{A}$ are C^*-weak Hopf algebras in duality.

The way from pseudo-multiplicative unitaries to multiplicative isometries is not completely understood. Although we show at the end of Section 6 that every U determines a multiplicative isometry V, unitalnes or regularity of this V remain unresolved.

2 Multiplicative partial isometries

Let $\mathcal{H}$ be a Hilbert space and $V:\mathcal{H}\otimes\mathcal{H}\to\mathcal{H}\otimes\mathcal{H}$ be a partial isometry, i.e. $VV^*V=V$. We shall say that V is a *multiplicative partial isometry* (MPI) if the following equations hold on the 3-fold tensor product $\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}$:

$$\begin{aligned} V_{23}V_{12} &= V_{12}V_{13}V_{23} && (2.1)\\ V_{13}V_{23}V_{23}^* &= V_{12}^*V_{12}V_{13} && (2.2)\\ V_{12}V_{12}^*V_{23} &= V_{23}V_{12}V_{12}^* && (2.3)\\ V_{12}V_{23}^*V_{23} &= V_{23}^*V_{23}V_{12}\ . && (2.4) \end{aligned}$$

The following equations are immediate consequences:

$$\begin{aligned} V_{12}^*V_{23}V_{12} &= V_{13}V_{23} && (2.5)\\ V_{23}V_{12}V_{23}^* &= V_{12}V_{13} && (2.6)\\ V_{12}V_{23}^* &= V_{23}^*V_{12}V_{13} && (2.7)\\ V_{12}^*V_{23} &= V_{13}V_{23}V_{12}^* && (2.8)\\ V_{12}V_{13}V_{13}^* &= V_{23}V_{23}^*V_{12} && (2.9)\\ V_{13}^*V_{13}V_{23} &= V_{23}V_{12}^*V_{12}\ . && (2.10) \end{aligned}$$

For example, in order to obtain (2.5) multiply (2.1) by V_{12}^* and then use (2.2). The reader may easily prove the remaining equations in order of appearence. For a geometrical interpretation of these equations see Section 6.

In this note we restrict ourselves to MPI's on finite dimensional $\mathcal{H}$. Let $\mathcal{L}(\mathcal{H})$ denote the space of linear operators on $\mathcal{H}$ and $\mathcal{L}(\mathcal{H})_*$ the space of linear functionals on $\mathcal{L}(\mathcal{H})$. Let V be any operator $V \in \mathcal{L}(\mathcal{H}) \otimes \mathcal{L}(\mathcal{H})$ and construct the linear maps

$$\lambda\colon \mathcal{L}(\mathcal{H})_* \to \mathcal{L}(\mathcal{H}) \qquad\qquad \rho\colon \mathcal{L}(\mathcal{H})_* \to \mathcal{L}(\mathcal{H}) \tag{2.11}$$

$$\lambda(\omega) := (\omega \otimes \mathrm{id}\,)(V) \qquad\qquad \rho(\omega) := (\mathrm{id}\, \otimes \omega)(V) \tag{2.12}$$

Their images $A := \lambda(\mathcal{L}(\mathcal{H})_*)$ and $\hat{A} := \rho(\mathcal{L}(\mathcal{H})_*)$, called the right leg and left leg of V, respectively, are subspaces of $\mathcal{L}(\mathcal{H})$ that are in duality with respect to the non-degenerate bilinear form

$$\langle \lambda(\omega), \rho(\omega') \rangle \;:=\; (\omega \otimes \omega')(V) \equiv \omega(\rho(\omega')) \equiv \omega'(\lambda(\omega)) \;. \tag{2.13}$$

One obtains directly that $V \in \hat{A} \otimes A$.

Let us introduce the following two binary operations on $\mathcal{L}(\mathcal{H})_*$.

$$\left.\begin{array}{rcl} (\omega \star \omega')(X) &:=& (\omega \otimes \omega')(V^*(\mathbf{1} \otimes X)V) \\ (\omega \diamond \omega')(X) &:=& (\omega \otimes \omega')(V(X \otimes \mathbf{1})V^*) \end{array}\right\} \quad X \in \mathcal{L}(\mathcal{H}) \tag{2.14}$$

If V is an MPI then we obtain

$$\lambda(\omega)\lambda(\omega') \;=\; (\omega \otimes \omega' \otimes \mathrm{id}\,)(V_{13}V_{23}) \overset{(2.5)}{=} \lambda(\omega \star \omega') \tag{2.15}$$

$$\rho(\omega)\rho(\omega') \;=\; (\mathrm{id}\, \otimes \omega \otimes \omega')(V_{12}V_{13}) \overset{(2.6)}{=} \rho(\omega \diamond \omega') \tag{2.16}$$

showing that A and $\hat{A}$ are subalgebras of $\mathcal{L}(\mathcal{H})$.

The next step is to introduce the would-be coproducts Δ and $\hat{\Delta}$, at first as linear maps $\mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}) \otimes \mathcal{L}(\mathcal{H})$,

$$\left.\begin{array}{rcl} \Delta(X) &:=& V(X \otimes \mathbf{1})V^* \\ \hat{\Delta}(X) &:=& V^*(\mathbf{1} \otimes X)V \end{array}\right\} \quad X \in \mathcal{L}(\mathcal{H}) \;. \tag{2.17}$$

Lemma 2.1 *Δ and $\hat{\Delta}$ restrict to algebra maps $\Delta\colon A \to A \otimes A$ and $\hat{\Delta}\colon \hat{A} \to \hat{A} \otimes \hat{A}$.*

Proof: The identities

$$\begin{array}{rcl} \Delta(\lambda(\omega)) &=& (\omega \otimes \mathrm{id}\, \otimes \mathrm{id}\,)(V_{23}V_{12}V_{23}^*) \overset{(2.6)}{=} (\omega \otimes \mathrm{id}\, \otimes \mathrm{id}\,)(V_{12}V_{13}) \;\in\; A \otimes A \\ \hat{\Delta}(\rho(\omega)) &=& (\mathrm{id}\, \otimes \mathrm{id}\, \otimes \omega)(V_{12}^*V_{23}V_{12}) \overset{(2.5)}{=} (\mathrm{id}\, \otimes \mathrm{id}\, \otimes \omega)(V_{13}V_{23}) \;\in\; \hat{A} \otimes \hat{A} \end{array}$$

show that $\Delta(A) \subset A \otimes A$ and $\hat{\Delta}(\hat{A}) \subset \hat{A} \otimes \hat{A}$ so we have the required restrictions. It remains to show multiplicativity of these restrictions.

$$\begin{aligned}
& \Delta(\lambda(\omega))\Delta(\lambda(\omega')) = (\omega \otimes \omega' \otimes \mathrm{id} \otimes \mathrm{id})(V_{13}V_{14}V_{23}V_{24}) \stackrel{(2.5)}{=} \\
= \; & (\omega \otimes \omega' \otimes \mathrm{id} \otimes \mathrm{id})(V_{12}^* V_{23} V_{12} V_{12}^* V_{24} V_{12}) \stackrel{(2.3)}{=} \\
= \; & (\omega \otimes \omega' \otimes \mathrm{id} \otimes \mathrm{id})(V_{12}^* V_{23} V_{24} V_{12}) = ((\omega \star \omega') \otimes \mathrm{id} \otimes \mathrm{id})(V_{12}V_{13}) \stackrel{(2.6)}{=} \\
= \; & \Delta(\lambda(\omega)\lambda(\omega'))
\end{aligned}$$

$$\begin{aligned}
& \hat{\Delta}(\rho(\omega))\hat{\Delta}(\rho(\omega')) = (\mathrm{id} \otimes \mathrm{id} \otimes \omega \otimes \omega')(V_{13}V_{23}V_{14}V_{24}) \stackrel{(2.6)}{=} \\
= \; & (\mathrm{id} \otimes \mathrm{id} \otimes \omega \otimes \omega')(V_{34}V_{13}V_{34}^* V_{34} V_{23} V_{34}^*) \stackrel{(2.4)}{=} \\
= \; & (\mathrm{id} \otimes \mathrm{id} \otimes \omega \otimes \omega')(V_{34}V_{13}V_{23}V_{34}^*) = (\mathrm{id} \otimes \mathrm{id} \otimes (\omega \diamond \omega'))(V_{13}V_{23}) \stackrel{(2.5)}{=} \\
= \; & \hat{\Delta}(\rho(\omega)\rho(\omega'))
\end{aligned}$$

Q.e.d.

From now on Δ and $\hat{\Delta}$ will denote these restrictions of the original maps (2.17).

Lemma 2.2 *Under the pairing $\langle \, , \, \rangle$ the comultiplication maps Δ and $\hat{\Delta}$ are the transposes of the multiplications on $\hat{A}$ and A, respectively. In particular Δ and $\hat{\Delta}$ are coassociative.*

Proof: We need to show that for $\omega, \omega', \omega'' \in \mathcal{L}(\mathcal{H})_*$

$$\begin{aligned}
\langle \lambda(\omega), \rho(\omega')\rho(\omega'') \rangle &= \langle \Delta(\lambda(\omega)), \rho(\omega') \otimes \rho(\omega'') \rangle \\
\langle \lambda(\omega)\lambda(\omega'), \rho(\omega'') \rangle &= \langle \lambda(\omega) \otimes \lambda(\omega'), \hat{\Delta}(\rho(\omega'')) \rangle
\end{aligned}$$

or, equivalently

$$\begin{aligned}
(\omega' \diamond \omega'')(\lambda(\omega)) &= (\omega \otimes \omega' \otimes \omega'')(V_{12}V_{13}) \\
(\omega \star \omega')(\rho(\omega'')) &= (\omega \otimes \omega' \otimes \omega'')(V_{13}V_{23})
\end{aligned}$$

which, up to an application of (2.6) or (2.5), are precisely the definitions of the convolution products (2.14). *Q.e.d.*

In this way we have shown that a multiplicative partial isometry determines a pair $(A, \hat{A})$ of algebras in duality such that the induced comultiplications are algebra maps. It is not clear, however, if these algebras have units or if they are closed under the *-operation. So we need further assumptions.

3 Unital MPI's and Weak Bialgebras

At first we will seek for the conditions on the finite dimensional MPI V that ensure that A and $\hat{A}$ are weak bialgebras (WBA's) in the sense of [3]. Obviously it is necessary that both of them should be unital algebras (hence counital coalgebras). We claim that this condition, called *unitalness*, is not only necessary but also sufficient. It is also shown that under this condition the elements of A and $\hat{A}$ realize a (not necessarily faithful) representation of the Weyl algebra (or Heisenberg double) $A \rtimes \hat{A}$ [2, 3].

Definition 3.1 *A finite dimensional MPI V on the Hilbert space $\mathcal{H}$ is* unital *if there exist functionals $\mathcal{L}(\mathcal{H})_* \ni \varepsilon$ and $\hat{\varepsilon}$ such that $A \ni \lambda(\hat{\varepsilon}) \equiv \mathbb{1}$ and $\hat{A} \ni \rho(\varepsilon) \equiv \hat{\mathbb{1}}$ are two-sided units for A and $\hat{A}$, respectively.*

In order to illustrate that, in contrast to multiplicative unitaries, finite dimensional MPI's are not always unital, let stand here a non-unital example. Let $\mathcal{H} = \mathbb{C}^2$ and define $V = e_{11} \otimes e_{12} + e_{22} \otimes e_{22}$ with a chosen set of *-matrix units $\{e_{ij}\}_{i,j\in\{1,2\}}$. Then one can see by inspection that V is an MPI, its left leg contains $\mathbb{1}$, but its right leg does not.

Although the functionals ε and $\hat{\varepsilon}$ in the above Definition are not unique they have a unique restriction onto A and $\hat{A}$, respectively. These restrictions (also denoted as ε and $\hat{\varepsilon}$) are then counits of A and $\hat{A}$, respectively.

If V is unital then A and $\hat{A}$ are WBA's provided the counits are weakly multiplicative or, equivalently, if the units $\hat{\mathbb{1}}$ and $\mathbb{1}$ are weakly comultiplicative. We show this latter property using

Lemma 3.2 *Let V be a finite dimensional unital MPI on the Hilbert space $\mathcal{H}$ with unit elements $\mathbb{1} \in A$ and $\hat{\mathbb{1}} \in \hat{A}$. Then*

$$\Delta(\mathbb{1}) = VV^* \tag{3.18}$$

$$\hat{\Delta}(\hat{\mathbb{1}}) = V^*V. \tag{3.19}$$

Proof: By (2.17) we have for any $\omega \in \mathcal{L}(\mathcal{H})_*$

$$\begin{aligned}(\mathbf{1} \otimes \lambda(\omega))\Delta(\mathbb{1}) &= (\omega \otimes \mathrm{id} \otimes \mathrm{id})(V_{13}V_{23}(\mathbf{1} \otimes \mathbb{1} \otimes \mathbf{1})V_{23}^*) \overset{(2.5)}{=} \\ &= (\omega \otimes \mathrm{id} \otimes \mathrm{id})(V_{12}^*V_{23}V_{12}(\mathbf{1} \otimes \mathbb{1} \otimes \mathbf{1})V_{23}^*).\end{aligned}$$

By the assumption that $\mathbb{1}$ is a (right) unit for A, $V(\mathbf{1} \otimes \mathbb{1}) = V$ and

$$(\mathbf{1} \otimes \lambda(\omega))\Delta(\mathbb{1}) = (\omega \otimes \mathrm{id} \otimes \mathrm{id})(V_{13}V_{23}V_{23}^*) = (\mathbf{1} \otimes \lambda(\omega))VV^*.$$

Setting $\omega = \hat{\varepsilon}$ and using the assumption that $\mathbb{1}$ is a (left) unit for A (3.18) is proven. A similar argument shows that

$$\hat{\Delta}(\hat{\mathbb{1}})(\rho(\omega) \otimes \mathbf{1}) = V^* V(\rho(\omega) \otimes \mathbf{1}) \tag{3.20}$$

for any $\omega \in \mathcal{L}(\mathcal{H})_*$, hence the substitution $\omega = \varepsilon$ proves (3.19). *Q.e.d.*

As a consequence of Lemma 3.2

$$(\Delta(\mathbb{1})\otimes\mathbb{1})(\mathbb{1}\otimes\Delta(\mathbb{1})) = V_{12}V_{12}^*V_{23}V_{23}^* \stackrel{(2.3)}{=} V_{23}V_{23}^*V_{12}V_{12}^* = (\mathbb{1}\otimes\Delta(\mathbb{1}))(\Delta(\mathbb{1})\otimes\mathbb{1}) \tag{3.21}$$

which, by (2.9), equals to

$$\mathbb{1}_{(1)} \otimes \mathbb{1}_{(2)} \otimes \mathbb{1}_{(3)} = V_{12}V_{13}V_{13}^*V_{12}^*. \tag{3.22}$$

Similarly,

$$(\hat{\Delta}(\hat{\mathbb{1}})\otimes\hat{\mathbb{1}})(\hat{\mathbb{1}}\otimes\hat{\Delta}(\hat{\mathbb{1}})) = V_{12}^*V_{12}V_{23}^*V_{23} \stackrel{(2.4)}{=} V_{23}^*V_{23}V_{12}^*V_{12} = (\hat{\mathbb{1}}\otimes\hat{\Delta}(\hat{\mathbb{1}}))(\hat{\Delta}(\hat{\mathbb{1}})\otimes\hat{\mathbb{1}}), \tag{3.23}$$

which, by (2.10), equals to

$$\hat{\mathbb{1}}_{(1)} \otimes \hat{\mathbb{1}}_{(2)} \otimes \hat{\mathbb{1}}_{(3)} = V_{23}^*V_{13}^*V_{13}V_{23}. \tag{3.24}$$

This proves that if V is unital then the resulting algebras A and $\hat{A}$ are WBA's in duality.

A further consequence of the above Lemma is that the subalgebras A^L and A^R of A, that were originally defined as the right leg and left leg, respectively of $\Delta(\mathbb{1})$ [3], appear now in the form

$$A^L = \{\, (\omega \otimes \mathrm{id}\,)(VV^*) \,|\, \omega \in \mathcal{L}(\mathcal{H})_* \,\} \tag{3.25}$$

$$A^R = \{\, (\mathrm{id}\, \otimes \omega)(VV^*) \,|\, \omega \in \mathcal{L}(\mathcal{H})_* \,\}\ . \tag{3.26}$$

Therefore they are selfadjoint subalgebras in $\mathcal{L}(\mathcal{H})$ even if we do not know whether A is selfadjoint. Similar conclusion holds for the subalgebras $\hat{A}^L$ and $\hat{A}^R$ of $\hat{A}$.

As far as the relative position of A and $\hat{A}$ in $\mathcal{L}(\mathcal{H})$ is concerned we want to show that A and $\hat{A}$ generate a representation of the Weyl algebra $A \rtimes \hat{A}$ on $\mathcal{H}$. As a matter of fact the pentagon equation (2.1) implies the commutation relation

$$\begin{aligned} \rho(\omega)\lambda(\omega') &= (\omega' \otimes \mathrm{id}\, \otimes \omega)(V_{23}V_{12}) = \\ &= (\omega' \otimes \mathrm{id}\, \otimes \omega)(V_{12}V_{13}V_{23}) = (\mathrm{id}\, \otimes \omega)(\Delta(\lambda(\omega'))V) = \\ &= \lambda(\omega')_{(1)}\langle \lambda(\omega')_{(2)}, \rho(\omega)_{(1)}\rangle \rho(\omega)_{(2)}. \end{aligned} \tag{3.27}$$

The only missing Weyl algebra relation is $\mathbb{1} = \hat{\mathbb{1}}$.

Proposition 3.3 *Let V be a finite dimensional unital MPI on the Hilbert space $\mathcal{H}$ with unit elements $\mathbb{1} \equiv \lambda(\hat{\varepsilon}) \in A$ and $\hat{\mathbb{1}} \equiv \rho(\varepsilon) \in \hat{A}$. Then*

$$\mathbb{1} = \hat{\mathbb{1}} \tag{3.28}$$

as elements of $\mathcal{L}(\mathcal{H})$.

Proof: We recall [3] that a projection from $\hat{A}$ onto $\hat{A}^L$ is provided by $\hat{\sqcap}^L(\varphi) := \hat{\varepsilon}(\hat{\mathbb{1}}_{(1)}\varphi)\hat{\mathbb{1}}_{(2)}$. Hence for arbitrary $\omega \in \mathcal{L}(\mathcal{H})_*$

$$\begin{aligned} \hat{\sqcap}^L(\rho(\omega)) &= (\hat{\varepsilon} \otimes \mathrm{id} \otimes \omega)(V_{12}^* V_{12} V_{13}) \stackrel{(2.2)}{=} (\hat{\varepsilon} \otimes \mathrm{id} \otimes \omega)(V_{13} V_{23} V_{23}^*) = \\ &= (\mathrm{id} \otimes \omega)((\mathbf{1} \otimes \mathbb{1})VV^*) = (\mathrm{id} \otimes \omega)(\Delta(\mathbb{1})). \end{aligned} \tag{3.29}$$

Setting $\omega = \varepsilon$ we obtain (3.28). *Q.e.d.*

As a byproduct equation (3.29) tells us that the subalgebras $A^R \subset A$ and $\hat{A}^L \subset \hat{A}$ coincide as subalgebras of $A \rtimes \hat{A}$ and therefore of $\mathcal{L}(\mathcal{H})$. As a counterpart of this relation one can also show that

$$\sqcap^R(\lambda(\omega)) \equiv \mathbb{1}_{(1)}\varepsilon(\lambda(\omega)\mathbb{1}_{(2)}) = (\omega \otimes \mathrm{id})(V^*V) = \langle \lambda(\omega), \hat{\mathbb{1}}_{(1)} \rangle \hat{\mathbb{1}}_{(2)}\ , \tag{3.30}$$

hence $A^R = \hat{A}^L$ and the identification is given by $A^R \ni x^R \mapsto (\hat{\mathbb{1}} \rightharpoonup x^R) \in \hat{A}^L$. This relation is called the amalgamation relation.

4 Regular MPI's and Weak Hopf Algebras

Given a finite dimensional unital MPI V and the associated WBA's A and $\hat{A}$ one may look for the extra conditions on V that ensure one of the following special cases to occur:

- There exist antipodes S and $\hat{S}$ making A and $\hat{A}$ weak Hopf algebras.
- A and $\hat{A}$ are closed under the $*$-operation.
- A and $\hat{A}$ are C^*-WHA's in duality.

It turns out that these cases occur at the same time. In this Section we give a necessary and sufficient condition for this to happen that is reminiscent to the regularity condition of [1].

Remark: Questions like whether A and $\hat{A}$ are selfadjoint do not occur in the works [1, 5]. In their approach the Hopf algebra (Hopf bimodule) is *defined* to be the selfadjoint closure of the right or left leg of the (pseudo-) multiplicative unitary. In our finite dimensional approach the WBA or WHA A is the right leg of the MPI V and not larger. On the one hand this is very natural in view of the duality of A and $\hat{A}$ under the pairing (2.13) but on the other hand this will cause difficulties if one wants to compare MPI's with pseudo-multiplicative unitaries (see Section 6).

Proposition 4.1 *Let V be a finite dimensional MPI on the Hilbert space $\mathcal{H}$ such that the resulting algebras A and $\hat{A}$ are WHA's with coproducts given in (2.17) and with (the unique) antipodes $S\colon A \to A$ and $\hat{S} : \hat{A} \to \hat{A}$. Then we have the relation*

$$V^* = (\hat{S} \otimes id)(V) \equiv (id \otimes S)(V) \tag{4.31}$$

and therefore A and $\hat{A}$ are *-subalgebras of $\mathcal{L}(\mathcal{H})$.

Proof: (2.13) implies that $V = \sum_i \beta^i \otimes b_i$ with any basis $\{b_i\}$ of A and its dual basis $\{\beta^i\}$ of $\hat{A}$. Let $V' := \sum_i \hat{S}(\beta^i) \otimes b_i \equiv \sum_i \beta^i \otimes S(b_i)$. We claim that $V' = V^*$. Using the assumption that A and $\hat{A}$ are WHA's in duality compute

$$\begin{aligned}
VV' &= \sum_{i,j} \beta^i \hat{S}(\beta^j) \otimes b_i b_j = \sum_k \sqcap^L(\beta^k) \otimes b_k = \\
&= \hat{1\!\!1}_{(2)} \otimes 1\!\!1 \leftharpoonup \hat{1\!\!1}_{(1)} = \hat{1\!\!1} \leftharpoonup 1\!\!1_{(1)} \otimes 1\!\!1_{(2)} = 1\!\!1_{(1)} \otimes 1\!\!1_{(2)} = VV^* \\
V'V &= \sum_{i,j} \hat{S}(\beta^i) \beta^j \otimes b_i b_j = \sum_k \sqcap^R(\beta^k) \otimes b_k = \\
&= \hat{1\!\!1}_{(1)} \otimes \hat{1\!\!1}_{(2)} \rightharpoonup 1\!\!1 = \hat{1\!\!1}_{(1)} \otimes \hat{1\!\!1}_{(2)} = V^*V
\end{aligned}$$

where in the last step of both cases we used the amalgamation relation (3.29-3.30). Now

$$\begin{aligned}
V^* &= V^*VV^* = V'VV^* = V'\Delta(1\!\!1) = \sum_i \beta^i 1\!\!1_{(1)} \otimes S(b_i) 1\!\!1_{(2)} = \\
&= \sum_i \beta^i \leftharpoonup 1\!\!1_{(1)} \otimes S(b_i) 1\!\!1_{(2)} = \beta^i \otimes S(b_i) S(1\!\!1_{(1)}) 1\!\!1_{(2)} = V'
\end{aligned}$$

therefore

$$\begin{aligned}
S(\lambda(\omega)) &= (\omega \otimes \mathrm{id}\,)(V^*) \\
\hat{S}(\rho(\omega)) &= (\mathrm{id}\, \otimes \omega)(V^*)
\end{aligned} \tag{4.32}$$

implying that $S(A) \subset A^*$ and $\hat{S}(\hat{A}) \subset \hat{A}^*$. This is possible for the bijections $S : A \to A$ and $\hat{S} : \hat{A} \to \hat{A}$ only if A and $\hat{A}$ are *-subalgebras of $\mathcal{L}(\mathcal{H})$. *Q.e.d.*

The next Proposition proves a converse result plus some more.

Proposition 4.2 *Suppose that the MPI V on the Hilbert space $\mathcal{H}$ is such that its right and left leg, A and $\hat{A}$, are *-subalgebras of $\mathcal{L}(\mathcal{H})$. Then V is unital and the expressions (4.32) define antipodes that make A and $\hat{A}$ C^*-WHA's in duality.*

Proof: Since *-subalgebras of $\mathcal{L}(\mathcal{H})$ are semisimple, A and $\hat{A}$ have units. Furthermore, being in duality by the pairing (2.13), they possess functionals ε and $\hat{\varepsilon}$ required in Definition 3.1. Thus V is unital and A and $\hat{A}$ are WBA's in duality by the results of Section 3.

In order to construct antipodes notice that if $\lambda(\omega) = 0$ then $\omega(\hat{A}^*) = \omega(\hat{A}) = 0$, therefore the S of (4.32) is a well defined map $A \to A$. Similarly, (4.32) defines a map $\hat{S}: \hat{A} \to \hat{A}$. These maps are the transpose of each other with respect to the canonical pairing (2.13),

$$\langle \hat{S}(\rho(\omega)), \lambda(\omega')\rangle = (\omega' \otimes \omega)(V^*) = \langle \rho(\omega), S(\lambda(\omega'))\rangle$$

for all $\omega, \omega' \in \mathcal{L}(\mathcal{H})_*$. It remained to show that the C^*-WHA axioms are satisfied.

Define the antilinear involution $_*: \mathcal{L}(\mathcal{H})_* \to \mathcal{L}(\mathcal{H})_*$ by $\omega_*(X) := \overline{\omega(X^*)}$ for $X \in \mathcal{L}(\mathcal{H})$. Then (4.32) can be rewritten as

$$\begin{aligned} S(\lambda(\omega)) &= \lambda(\omega_*)^* \\ \hat{S}(\rho(\omega)) &= \rho(\omega_*)^* \, . \end{aligned}$$

By showing that $_*$ preserves both convolution products (2.14),

$$(\omega \star \omega')_* = \omega_* \star \omega'_* \ , \quad (\omega \diamond \omega')_* = \omega_* \diamond \omega'_* \ ,$$

we find that both S and $\hat{S}$ are anti-multiplicative and anti-comultiplicative. Finally

$$\begin{aligned} \lambda(\omega)_{(1)} \otimes \lambda(\omega)_{(2)} S(\lambda(\omega)_{(3)}) &= (\omega \otimes \mathrm{id} \otimes \mathrm{id})(V_{12}V_{13}V_{13}^*) \stackrel{(2.9)}{=} \\ &= (\omega \otimes \mathrm{id} \otimes \mathrm{id})(V_{23}V_{23}^*V_{12}) = \Delta(\mathbb{1})(\lambda(\omega) \otimes \mathbf{1}) \\ \rho(\omega)_{(1)} \otimes \rho(\omega)_{(2)} \hat{S}(\rho(\omega)_{(3)}) &= (\mathrm{id} \otimes \mathrm{id} \otimes \omega)(V_{13}V_{23}V_{23}^*) \stackrel{(2.2)}{=} \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \omega)(V_{12}^*V_{12}V_{13}) = \hat{\Delta}(\hat{\mathbb{1}})(\rho(\omega) \otimes \mathbf{1}) \end{aligned}$$

prove that the WHA axioms of [12] hold both in A and $\hat{A}$. Since the coproducts (2.17) are manifestly *- algebra maps, A and $\hat{A}$ are *-WHA's. Furthermore the defining representations of A and $\hat{A}$ on $\mathcal{H}$ are faithful *-representations by construction therefore A and $\hat{A}$ are C^*-WHA's. *Q.e.d.*

It remains to characterize the situation of A and $\hat{A}$ being selfadjoint in "more algebraic" terms, i.e. using only the relative positions of A and $\hat{A}$ in $\mathcal{L}(\mathcal{H})$ without referring to their *-structure. This will be the regularity condition on the multiplicative isometry V.

In analogy with [1] we define the subspace $\mathcal{C}(V) := V_2\mathcal{L}(\mathcal{H})V_1$ in $\mathcal{L}(\mathcal{H})$, where $V_1 \otimes V_2$ stands for V, and verify using the pentagon equation (2.1) that $\mathcal{C}(V)$ is a subalgebra of $\mathcal{L}(\mathcal{H})$.

Lemma 4.3 *Let V be a unital MPI on the Hilbert space $\mathcal{H}$. If $\mathcal{C}(V)$ is a *-subalgebra of $\mathcal{L}(\mathcal{H})$ then so are A and $\hat{A}$.*

Proof: The proof generalizes the one of Proposition 3.5 in [1]. At first we show that

$$A^* = \{(\omega \otimes \omega' \otimes \mathrm{id}\,)(\Sigma_{12}V_{23}^*V_{12}V_{13}) | \omega, \omega' \in \mathcal{L}(\mathcal{H})_*\} \tag{4.33}$$

where $\Sigma : \mathcal{H}\otimes\mathcal{H} \to \mathcal{H}\otimes\mathcal{H}$ is the flip map. This follows from the computations

$$\begin{aligned}(\omega \otimes \omega' \otimes \mathrm{id}\,)(\Sigma_{12}V_{23}^*V_{12}V_{13}) &\overset{(2.7)}{=} (\omega \otimes \omega' \otimes \mathrm{id}\,)(\Sigma_{12}V_{12}V_{23}^*) = \\ &= (\omega' \otimes \mathrm{id}\,)(((\omega \otimes \mathrm{id}\,)(\Sigma V) \otimes \mathbf{1})V^*) \in A^*\end{aligned}$$

and

$$(\omega \otimes \mathrm{id}\,)(V^*) = (\omega_1 \otimes \omega_2 \otimes \mathrm{id}\,)(\Sigma_{12}V_{23}^*V_{12}V_{13})$$

where we introduced $\omega_1 \otimes \omega_2 \in \mathcal{L}(\mathcal{H})_* \otimes \mathcal{L}(\mathcal{H})_*$ by setting $(\omega_1 \otimes \omega_2)(X) := (\hat{\varepsilon} \otimes \omega)(\Sigma X)$. Thus (4.33) is proven. The next identity

$$\begin{aligned}(\omega \otimes \omega' \otimes \mathrm{id}\,)(\Sigma_{12}V_{23}^*V_{12}V_{13}) &= (\omega \otimes \omega' \otimes \mathrm{id}\,)(V_{13}^*\Sigma_{12}V_{12}V_{13}) = \\ &= (\omega \otimes \mathrm{id}\,)(V^*((\mathrm{id}\, \otimes \omega')(\Sigma V) \otimes \mathbf{1})V)\end{aligned}$$

shows that if $\mathcal{C}(V) \equiv \{(id\otimes\omega)(\Sigma V) | \omega \in \mathcal{L}(\mathcal{H})_*\}$ is closed under the *-operation then so is A^* hence A.

In the case of $\hat{A}$ repeat the above argument using the fact that in passing from the MPI V to the MPI $\Sigma V^*\Sigma$ the left leg $\hat{A}(V)$ becomes the adjoint of the right leg $A(\Sigma V^*\Sigma)$ and also $\mathcal{C}(\Sigma V^*\Sigma) = \mathcal{C}(V)^*$. *Q.e.d.*

A^R is the subalgebra of $\mathcal{L}(\mathcal{H})$ spanned by the elements $\{(\omega\otimes \mathrm{id}\,)(V^*V)\,|\,\omega\in \mathcal{L}(\mathcal{H})_*\,\}$. It is obviously a *-subalgebra and for $a^R=(\omega\otimes\mathrm{id}\,)(V^*V)$

$$\begin{aligned}(\mathbf{1}\otimes a^R)V &= (\omega\otimes\mathrm{id}\,\otimes\mathrm{id}\,)(V_{13}^*V_{13}V_{23}) \stackrel{(2.10)}{=} \\ &= (\omega\otimes\mathrm{id}\,\otimes\mathrm{id}\,)(V_{23}V_{12}^*V_{12}) = V(a^R\otimes\mathbf{1})\end{aligned}$$

hence A^R commutes with $\mathcal{C}(V)$. Let us make the following

Definition 4.4 *A finite dimensional unital MPI V on the Hilbert space $\mathcal{H}$ is called* regular *if*

$$\mathcal{C}(V)=(A^R)'\cap \mathbb{1}\mathcal{L}(\mathcal{H})\mathbb{1}\ . \tag{4.34}$$

In the special case of V being a multiplicative unitary the A^R consists only of the scalars therefore $(A^R)'\cap\mathbb{1}\mathcal{L}(\mathcal{H})\mathbb{1}=\mathcal{L}(\mathcal{H})$ and our regularity condition reduces to the regularity of [1]. Although in finite dimensions all multiplicative unitaries are regular by Theorem 4.10 of [1] we do not know any generalization of this result to multiplicative isometries.

Theorem 4.5 *The algebras A and $\hat{A}$ obtained from a finite dimensional MPI $(V,\mathcal{H})$ are *-subalgebras of $\mathcal{L}(\mathcal{H})$ if and only if V is unital and regular.*

Proof: Since A^R is a *-subalgebra of $\mathcal{L}(\mathcal{H})$ so is its commutant. This implies that if V is unital and regular then $\mathcal{C}(V)$ is a *-subalgebra of $\mathcal{L}(\mathcal{H})$ and using Lemma 4.3 the *if* part follows.

To prove the converse statement suppose that A and $\hat{A}$ are *-subalgebras of $\mathcal{L}(\mathcal{H})$ so they are C^*-WHA's in duality by Proposition 4.2. Then V is necessarily unital. In this case A^R is the right subalgebra of A coinciding with the left subalgebra of $\hat{A}$ (see (3.29)).

Knowing already that $\mathcal{C}(V)\subset(A^R)'\cap\mathbb{1}\mathcal{L}(\mathcal{H})\mathbb{1}$ it remains to show that also $(A^R)'\cap\mathbb{1}\mathcal{L}(\mathcal{H})\mathbb{1}\subset\mathcal{C}(V)$. For that purpose let $X\in(A^R)'\cap\mathbb{1}\mathcal{L}(\mathcal{H})\mathbb{1}$. Then

$$\begin{aligned}X &= X\mathbb{1}=X\mathbb{1}_{(1)}S(\mathbb{1}_{(2)})=X\mathbb{1}_{(1)}\hat{S}^{-1}(\mathbb{1}_{(2)}\rightharpoonup\hat{\mathbb{1}})= \\ &= X\sum_k \sqcap^R(b_k)\hat{S}^{-1}(\beta^k)=\sum_k\sqcap^R(b_k)X\hat{S}^{-1}(\beta^k)= \\ &= \sum_k S(b_{k(1)})b_{k(2)}X\hat{S}^{-1}(\beta^k)=\sum_{i,j}S(b_i)b_jX\hat{S}^{-1}(\beta^i\beta^j)= \\ &= \sum_{i,j}b_ib_jX\hat{S}^{-1}(\beta^j)\beta^i\in\mathcal{C}(V)\end{aligned}$$

finishes the proof. *Q.e.d.*

With the above Theorem we have characterized the class of MPI's that lead to C^*-WHA's. The question arises whether all C^*-WHA's can be obtained in this way. The answer is in fact very easy. Let A be a C^*-weak Hopf algebra and let $\pi\colon A \rtimes \hat{A} \to \mathcal{L}(\mathcal{H})$ be a *-representation such that the restrictions $\pi|_A$ and $\pi|_{\hat{A}}$ are faithful. Choose a basis $\{b_i\}$ of A and construct the dual basis $\{\beta^i\}$, $\langle b_i, \beta^j \rangle = \delta_{ij}$ of $\hat{A}$. Then

$$V \;:=\; \sum_i \,\pi(\beta^i) \,\otimes\, \pi(b_i) \tag{4.35}$$

is a multiplicative partial isometry in the sense of (2.1), (2.2), (2.3), and (2.4) and furthermore it is unital and regular in the sense of Definitions 3.1 and 4.4. The proof of this statement is an elementary weak Hopf calculus which we omit. Notice that as a special case we obtain the "classical" example when $\mathcal{H}$ is the left regular representation of a C^*-WHA A with scalar product provided by the Haar measure, $(x, y) = \langle x^*y, \hat{h} \rangle$, $x, y \in A$. In this case the action of V is given by $V(x \otimes y) = x_{(1)} \otimes x_{(2)}y$.

5 Pseudo-multiplicative unitaries in finite dimensions

In order to discuss the relation of MPI's to the pseudo-multiplicative unitaries [13, 5] we specialize their definition to the case when the Hilbert space in the game is finite dimensional. At first we exhibit the Connes-Sauvageot relative tensor product [11] of finite dimensional modules as a subspace in the ordinary tensor product. Then the pseudo-multiplicative unitary U will be obtained by restricting the domain and range of the MPI V to its initial and final support. It should be emphasized, however, that the pseudo-multiplicative unitary has to be supplied with an a priori knowledge of the algebra A^L and a faithful state on it while this information is implicitely stored in the structure of V.

Let B be a finite dimensional C^*-algebra, $\mathcal{H}$ and $\mathcal{K}$ finite dimensional Hilbert spaces, $\mathcal{H}$ carrying a right and $\mathcal{K}$ a left B-module structure, i.e. there are given *-homomorphisms $\beta : B^o \to \mathcal{L}(\mathcal{H})$ and $\gamma : B \to \mathcal{L}(\mathcal{K})$. If $\psi : B \to \mathbb{C}$ is a faithful positive linear functional the *relative tensor product* of $\mathcal{H}_\beta$ and ${}_\gamma\mathcal{K}$ over ψ is defined to be the subspace in $\mathcal{H} \otimes \mathcal{K}$ obtained as the image of a projection $E_\psi \in \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ constructed below.

Let $\{a_i\}$ be a basis of B and $\{b_i\}$ the dual basis with respect to ψ, i.e. $\psi(b_i a_j) = \delta_{i,j}$. Then $x = \sum_i a_i \psi(b_i x)$ for all $x \in B$, i.e. $\{a_i, b_i\}$ is a *quasibasis*

of ψ in the sense of [14]. The *index* of ψ, $\lambda := \sum_i a_i b_i$, is a positive invertible element of Center B. The *modular automorphism* of ψ is the (non-*) automorphism θ_ψ of B satisfying $\psi(xy) = \psi(y\theta_\psi(x))$ for all $x, y \in B$. In terms of these data we can define an element $e_\psi \in B^o \otimes B$ by the formula

$$e_\psi \equiv \sum_i u_i \otimes v_i := \sum_i \lambda^{-1} a_i \otimes \theta_\psi^{1/2}(b_i) \ . \tag{5.36}$$

Checking that e_ψ is a Hermitean idempotent we have $E_\psi := (\beta \otimes \gamma)(e_\psi)$ as the projection defining the relative tensor product $\mathcal{H} \underset{\psi}{\otimes} \mathcal{K} := E_\psi(\mathcal{H} \otimes \mathcal{K})$. The image of $\xi \otimes \eta \in \mathcal{H} \otimes \mathcal{K}$ in the relative tensor product will be denoted by $\xi \underset{\psi}{\otimes} \eta$. Using the property $\sum_i a_i \otimes b_i x = \sum_i x a_i \otimes b_i$, $x \in B$, of the quasibasis we immediately obtain the amalgamation relation

$$\xi \underset{\psi}{\otimes} \gamma(x)\eta = \beta \circ \theta_\psi^{-1/2}(x) \xi \underset{\psi}{\otimes} \eta \tag{5.37}$$

for all $\xi \underset{\psi}{\otimes} \eta \in \mathcal{H} \underset{\psi}{\otimes} \mathcal{K}$.

The above definition of the relative tensor product applies also to $\mathcal{K} \underset{\psi^o}{\otimes} \mathcal{H}$ if we replace B with B^o and call the resulting functional ψ^o. The identities $\sum_i b_i \psi(x a_i) = x$ and $\psi(yx) = \psi({\theta_\psi}^{-1}(x) y)$ show that $\sum_i a_i^o \otimes b_i^o := \sum_i b_i \otimes a_i$ is the quasi-basis of ψ^o and $\theta_{\psi^o} = \theta_\psi^{-1}$. Therefore $\sum_i u_i^o \otimes v_i^o = \sum_i u_i \otimes v_i$ and $\mathcal{K} \underset{\psi^o}{\otimes} \mathcal{H}$ is defined by the projection $E_{\psi^o} = (\gamma \otimes \beta)(e_\psi)$. Denoting the image of $\eta \otimes \xi$ in $\mathcal{K} \underset{\psi^o}{\otimes} \mathcal{H}$ by $\eta \underset{\psi^o}{\otimes} \xi$, we obtain the amalgamation

$$\eta \underset{\psi^o}{\otimes} \beta(x) \xi = \gamma \circ \theta_\psi^{\frac{1}{2}}(x) \eta \underset{\psi^o}{\otimes} \xi \tag{5.38}$$

for all $\eta \underset{\psi^o}{\otimes} \xi \in \mathcal{K} \underset{\psi^o}{\otimes} \mathcal{H}$.

Some caution is in order with the equations (5.37) and (5.38). They must not be considered as 'the operator $\mathbf{1} \otimes \beta(x)$', ...etc, acting on $\eta \underset{\psi^o}{\otimes} \xi$. Rather the vectors $\eta \otimes \beta(x)\xi \in \mathcal{K} \otimes \mathcal{H}$,...etc, are mapped into the subspace $\mathcal{K} \underset{\psi^o}{\otimes} \mathcal{H}$. Only operators $X \in \mathcal{L}(\mathcal{K})$ commuting with $\gamma(B)$ and $Y \in \mathcal{L}(\mathcal{H})$ commuting with $\beta(B)$ can be restricted to operators $(\mathbf{1}_\mathcal{H} \underset{\psi}{\otimes} X)$, $(Y \underset{\psi}{\otimes} \mathbf{1}_\mathcal{K}) \in \mathcal{L}(\mathcal{H} \underset{\psi}{\otimes} \mathcal{K})$ and $(\mathbf{1}_\mathcal{K} \underset{\psi^o}{\otimes} Y)$, $(X \underset{\psi^o}{\otimes} \mathbf{1}_\mathcal{H}) \in \mathcal{L}(\mathcal{K} \underset{\psi^o}{\otimes} \mathcal{H})$.

For later convenience we supress the letters β and γ and write $\xi \cdot b$ and $b \cdot \eta$ for $\beta(b)\xi$ and $\gamma(b)\eta$, respectively. In this spirit we may think $\underset{\psi}{\otimes}$ as the symbol $\cdot u_i \otimes v_i \cdot$ (with the i summed over).

The usual flip operator $\Sigma: \mathcal{H} \otimes \mathcal{K} \to \mathcal{K} \otimes \mathcal{H}$ determines an isomorphism $\Sigma_\psi: \mathcal{H} \underset{\psi}{\otimes} \mathcal{K} \to \mathcal{K} \underset{\psi^o}{\otimes} \mathcal{H}$ by restriction since Σ intertwines between the projections E_ψ and E_{ψ^o}, or in other words, because $\sum_i u_i \otimes v_i = \sum_i v_i \otimes u_i$. This follows using the fact that the modular automorphism is necessarily inner on a finite dimensional C^*-algebra. As a matter of fact let $g_\psi \in B$ be a positive element implementing θ_ψ, i.e. $g_\psi x g_\psi^{-1} = \theta_\psi(x)$ for all $x \in B$. Then

$$\begin{aligned}\sum_i v_i \otimes u_i &= \lambda^{-1} \sum_i g_\psi^{\frac{1}{2}} b_i g_\psi^{-\frac{1}{2}} \otimes a_i = \lambda^{-1} \sum_i g_\psi^{-\frac{1}{2}} \theta_\psi(b_i) g_\psi^{\frac{1}{2}} \otimes a_i = \\ &= \lambda^{-1} \sum_i g_\psi^{-\frac{1}{2}} a_i g_\psi^{\frac{1}{2}} \otimes b_i = \sum_i u_i \otimes v_i.\end{aligned}$$

With the above method one can construct also *multiple* relative tensor products of modules over (different) finite dimensional C^*-algebras. Let A and B be finite dimensional C^*-algebras, $\mathcal{H}, \mathcal{K}$ and $\mathcal{M}$ Hilbert spaces with the following module structures: $\mathcal{H}$ be a right A-module, $\mathcal{M}$ an A-B bimodule, and $\mathcal{K}$ a left B-module. Let $\phi : A \to \mathbb{C}$ and $\psi : B \to \mathbb{C}$ be faithful positive linear functionals. Then there are two threefold relative tensor products defined respectively by the formulae

$$\begin{aligned}\mathcal{H} \underset{\phi}{\otimes} (\mathcal{M} \underset{\psi}{\otimes} \mathcal{K}) &:= \sum_{i,j} \mathcal{H} \cdot u_i^\phi \otimes v_i^\phi \cdot (\mathcal{M} \cdot u_j^\psi \otimes v_j^\psi \cdot \mathcal{K}) \\ (\mathcal{H} \underset{\phi}{\otimes} \mathcal{M}) \underset{\psi}{\otimes} \mathcal{K} &:= \sum_{i,j} (\mathcal{H} \cdot u_i^\phi \otimes v_i^\phi \cdot \mathcal{M}) \cdot u_j^\psi \otimes v_j^\psi \cdot \mathcal{K},\end{aligned}$$

which, as subspaces of $\mathcal{H} \otimes \mathcal{M} \otimes \mathcal{K}$, coincide up to the associativity natural isomorphism in the category of Hilbert spaces. Supressing this natural isomorphism we can denote this Hilbert space by $\mathcal{H} \underset{\phi}{\otimes} \mathcal{M} \underset{\psi}{\otimes} \mathcal{K}$.

Considering the A-actions on $\mathcal{H}$ and $\mathcal{M}$ as A^o-actions we have also the Hilbert space $(\mathcal{M} \underset{\psi}{\otimes} \mathcal{K}) \underset{\phi^o}{\otimes} \mathcal{H}$. Similarly one can define $\mathcal{K} \underset{\psi^o}{\otimes} (\mathcal{H} \underset{\phi}{\otimes} \mathcal{M})$ and $\mathcal{K} \underset{\psi^o}{\otimes} \mathcal{M} \underset{\phi^o}{\otimes} \mathcal{H}$. They are all naturally isomorphic under the flip maps:

$$
\begin{tikzcd}
\mathcal{H} \underset{\phi}{\otimes} \mathcal{M} \underset{\psi}{\otimes} \mathcal{K} \arrow[r, "\Sigma_\phi"] \arrow[d, "\Sigma_\psi"'] & (\mathcal{M} \underset{\psi}{\otimes} \mathcal{K}) \underset{\phi^o}{\otimes} \mathcal{H} \arrow[d, "\Sigma_\psi \underset{\phi^o}{\otimes} \mathbf{1}_{\mathcal{H}}"] \\
\mathcal{K} \underset{\psi^o}{\otimes} (\mathcal{H} \underset{\phi}{\otimes} \mathcal{M}) \arrow[r, "\mathbf{1}_{\mathcal{K}} \underset{\psi^o}{\otimes} \Sigma_\phi"'] & \mathcal{K} \underset{\psi^o}{\otimes} \mathcal{M} \underset{\phi^o}{\otimes} \mathcal{H}
\end{tikzcd}
$$

6 The relation of U and V

In this Section we will present two constructions. At first we show how a finite dimensional unital multiplicative isometry $(V, \mathcal{H})$ determines a pseudo-multiplicative unitary U. After that starting from a finite dimensional pseudo-multiplicative unitary U we construct a MPI V.

Let V be a unital MPI on the finite dimensional Hilbert space $\mathcal{H}$, and A, $\hat{A}$ the associated WBA's in duality, both acting on $\mathcal{H}$. By Lemma 3.2 the left and right subalgebras of A and of $\hat{A}$ are selfadjoint subalgebras of $\mathcal{L}(\mathcal{H})$. In particular A^L is a C^*-algebra and the counit ε restricts to a faithful positive functional on A^L.

We need the following facts from the theory of weak bialgebras [8, 3]. Although an antipode may not exist on A we can define a would-be-antipode S on the subalgebra $A^L A^R$ by setting $S(x^L x^R) := \sqcap^L(x^R)\sqcap^R(x^L) \equiv (\hat{\mathbb{1}} \leftharpoonup x^L) \rightharpoonup \mathbb{1} \leftharpoonup (x^R \rightharpoonup \hat{\mathbb{1}})$. Then the element $S(\mathbb{1}_{(1)}) \otimes \mathbb{1}_{(2)} = \mathbb{1}_{(2)} \otimes S^{-1}(\mathbb{1}_{(1)}) \in A^L \otimes A^L$ provides a quasibasis of $\varepsilon \colon A^L \to \mathbb{C}$, hence $\varepsilon|_{A^L}$ has index $\mathbb{1}$. The modular automorphism of $\varepsilon|_{A^L}$ is $\theta = S^2|_{A^L}$ and it is implemented by a positive element $g_L \in A^L$. Although g_L is not unique, the formulae $\theta^{1/2}(x^L) = g_L^{1/2} x^L g_L^{-1/2}$ and $S_o := S \circ \theta^{-1/2}$ do not depend on this ambiguity. Here S_o is a "unitary antipode" satisfying $S_o \circ {}^* = {}^* \circ S_o$ and $S_o^2 = \mathrm{id}$. By means of these definitions we can construct

$$e_\varepsilon := \mathbb{1}_{(2)} \otimes \theta^{1/2}(S^{-1}(\mathbb{1}_{(1)})) \tag{6.39}$$

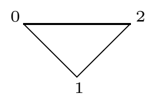
which is precisely the Hermitean idempotent (5.36) needed in relative tensor products of A^L-modules over ε or ε^o.

Corresponding to the three C^*-subalgebras A^L, $A^R \equiv \hat{A}^L$, and $\hat{A}^R$ of $\mathcal{L}(\mathcal{H})$ there are three mutually commuting actions of A^L on $\mathcal{H}$:

$$\alpha_{01}(x^L)\xi := x^L\xi \qquad \alpha_{02}(x^L)\xi := S_o(x^L)\xi \qquad \alpha_{12}(x^L)\xi := (x^L \rightharpoonup \hat{\mathbb{1}})\xi \tag{6.40}$$

α_{01} and α_{12} are left actions while α_{02} is a right action. It is tempting to visualize this trimodule structure of $\mathcal{H}$ by drawing a triangle (012) for the Hilbert space

$$\mathcal{H} \ = \tag{6.41}$$



and say that the edge (ij) is a left or right action of A^L according to whether the relative orientation of (ij) to the 2-simplex (012) is positive or negative.

Now we want to exhibit the source and target spaces of the partial isometry V as relative tensor products of $\mathcal{H}$ with itself. For that purpose we compute

$$\begin{aligned}
V^*V &= \hat{\Delta}(\hat{\mathbb{1}}) = \hat{\mathbb{1}}_{(1)} \otimes \hat{\mathbb{1}}_{(2)} \rightharpoonup \mathbb{1} = \mathbb{1}_{(2)} \rightharpoonup \hat{\mathbb{1}} \otimes \mathbb{1}_{(1)} = \\
&= \mathbb{1}_{(2)} \rightharpoonup \hat{\mathbb{1}} \otimes S_o(g_L^{\frac{1}{2}} S^{-1}(\mathbb{1}_{(1)}) g_L^{-\frac{1}{2}}) = (\alpha_{12} \otimes \alpha_{02})(e_\varepsilon)\ , \\
VV^* &= \Delta(\mathbb{1}) = S_o(g_L^{\frac{1}{2}} S^{-1}(\mathbb{1}_{(1)}) g_L^{-\frac{1}{2}}) \otimes \mathbb{1}_{(2)} = \\
&= (\alpha_{02} \otimes \alpha_{01})(e_\varepsilon)\ .
\end{aligned}$$

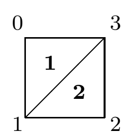
This means that we may identify the source and the target spaces of V with the following relative tensor products:

$$V^*V(\mathcal{H} \otimes \mathcal{H}) \ = \ \mathcal{H}_{\alpha_{12}} \underset{\varepsilon^o}{\otimes} {}_{\alpha_{02}}\mathcal{H} \tag{6.42}$$

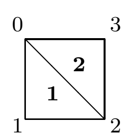
$$VV^*(\mathcal{H} \otimes \mathcal{H}) \ = \ \mathcal{H}_{\alpha_{02}} \underset{\varepsilon}{\otimes} {}_{\alpha_{01}}\mathcal{H}. \tag{6.43}$$

As a graphical representation of these relative tensor products one draws two triangles glued together along the edges corresponding to the amalgamated actions:

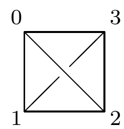
$$\mathcal{H}_{\alpha_{12}} \underset{\varepsilon^o}{\otimes} {}_{\alpha_{02}}\mathcal{H} \ = \tag{6.44}$$



$$\mathcal{H}_{\alpha_{02}} \underset{\varepsilon}{\otimes} {}_{\alpha_{01}}\mathcal{H} \ = \tag{6.45}$$



The numbering of the faces refer to their order in the tensor product. An other suggestive notation would be to denote the domain of V by $\mathcal{H}\boxslash\mathcal{H}$ and its range by $\mathcal{H}\boxbslash\mathcal{H}$. We can now define the operator $U\colon\mathcal{H}\boxslash\mathcal{H}\to\mathcal{H}\boxbslash\mathcal{H}$ as the restriction of V to its domain and range. The natural representation of this operator is then the tetrahedron

$U \;=$ 

(6.46)

or, better to say, this projection of the tetrahedron. Namely, the "equator" $\{(01),(12),(23),(30)\}$ is distinguished by dividing the surface into a "Northern hemisphere" $\{(012),(023)\}$ and a "Southern hemisphere" $\{(013),(123)\}$ corresponding to the range and domain of U, respectively.

Both the range and domain of U are quadrimodules, i.e. A^L acts on them via 3 left actions $\alpha_{01},\alpha_{12},\alpha_{23}$ and 1 right action α_{03}, and these 4 perimeter actions commute with each other. For example α_{12} acts on $\mathcal{H}\boxslash\mathcal{H}$ as $\mathrm{id}\otimes\alpha_{01}$ and on $\mathcal{H}\boxbslash\mathcal{H}$ as $\alpha_{12}\otimes\mathrm{id}$. Now U can be shown to intertwine these four actions, $\alpha_{ij}(x^L)U=U\alpha_{ij}(x^L)$, $x^L\in A^L$, $(ij)=(01),(12),(23),(03)$. The intertwiner relations are consequences of the following identities for V:

$$V(x^L\otimes\mathbf{1}) \stackrel{(01)}{=} (x^L\otimes\mathbf{1})V \qquad x^L\in A^L \tag{6.47a}$$

$$V(\mathbf{1}\otimes x^L) \stackrel{(12)}{=} (\varphi^R\otimes\mathbf{1})V \qquad \varphi^R=x^L\rightharpoonup\hat{\mathbb{1}},\ x^L\in A^L \tag{6.47b}$$

$$V(\mathbf{1}\otimes\varphi^R) \stackrel{(23)}{=} (\mathbf{1}\otimes\varphi^R)V \qquad \varphi^R\in\hat{A}^R \tag{6.47c}$$

$$V(x^R\otimes\mathbf{1}) \stackrel{(03)}{=} (\mathbf{1}\otimes x^R)V \qquad x^R\in A^R \tag{6.47d}$$

(Here $\stackrel{(ij)}{=}$ refers to the edge (ij) of the tetrahedron (0123) and not to an equation number as before.) The intertwiner relations for U are precisely the four equations in Definition 5.6.i of [5]. Thus, in order to see that our U is a pseudo-multiplicative unitary, we are left with showing that U satisfies the pentagon equation of Figure 1. Before doing that we remark that on the remaining two edges of the tetrahedron we have the amalgamation relations (5.38) and (5.37),

$$\xi\underset{\varepsilon^o}{\otimes}\alpha_{02}(x^L)\eta \;=\; \alpha_{12}(g_L^{\frac12}x^Lg_L^{-\frac12})\xi\underset{\varepsilon^o}{\otimes}\eta \tag{6.48}$$

$$\xi\underset{\varepsilon}{\otimes}\alpha_{01}(x^L)\eta \;=\; \alpha_{02}(g_L^{-\frac12}x^Lg_L^{\frac12})\xi\underset{\varepsilon}{\otimes}\eta\;, \tag{6.49}$$

$$
\begin{tikzcd}
\binom{04}{1}\underset{14}{\otimes}\binom{14}{2}\underset{24}{\otimes}\binom{24}{3} \arrow[rr, "{U\binom{04}{12}\underset{24}{\otimes}1\binom{24}{3}}"] \arrow[d, "{1\binom{04}{1}\underset{14}{\otimes}U\binom{14}{23}}"] & & \binom{02}{1}\underset{02}{\otimes}\binom{04}{2}\underset{24}{\otimes}\binom{24}{3} \arrow[r, "{1\binom{02}{1}\underset{02}{\otimes}U\binom{04}{23}}"] & \binom{02}{1}\underset{02}{\otimes}\binom{03}{2}\underset{03}{\otimes}\binom{04}{3} \\
\binom{04}{1}\underset{14}{\otimes}[\binom{13}{2}\underset{13}{\otimes}\binom{14}{3}] \arrow[dr, "{1\binom{04}{1}\underset{14}{\otimes}\Sigma(\mathbf{2},\mathbf{3})}"] & & & \\
& \binom{04}{1}\underset{14}{\otimes}\binom{14}{3}\underset{13}{\otimes}\binom{13}{2} \arrow[dl, "{\Sigma(\mathbf{12},\mathbf{3})}"] & & \\
\binom{13}{2}\underset{13}{\otimes}[\binom{04}{1}\underset{14}{\otimes}\binom{14}{3}] \arrow[d, "{1\binom{13}{2}\underset{13}{\otimes}U\binom{04}{13}}"] & & & \\
\binom{13}{2}\underset{13}{\otimes}\binom{03}{1}\underset{03}{\otimes}\binom{04}{3} \arrow[dr, "{\Sigma(\mathbf{1},\mathbf{2})\underset{03}{\otimes}1\binom{04}{3}}"] & & & \\
& [\binom{03}{1}\underset{13}{\otimes}\binom{13}{2}]\underset{03}{\otimes}\binom{04}{3} \arrow[uuuuurr, "{U\binom{03}{12}\underset{03}{\otimes}1\binom{04}{3}}"'] & &
\end{tikzcd}
$$

Figure 1: The ‘pentagon’ equation for the pseudo-multiplicative unitary U.

which have their origin in the V-identities

$$V(\varphi^R \otimes \mathbf{1}) = V(\mathbf{1} \otimes x^R) \qquad \varphi^R = x^R \rightharpoonup \hat{\mathbb{1}} \tag{6.50}$$
$$(\varphi^L \otimes \mathbf{1})V = (\mathbf{1} \otimes x^L)V \qquad \varphi^L = \hat{\mathbb{1}} \leftharpoonup x^L \tag{6.51}$$

As for the pentagon equation is concerned we need a more concise notation for multiple relative tensor products. Therefore we use the symbol $\binom{ik}{j}$ to denote a copy of $\mathcal{H}$ associated to the triangle (ijk). The symbol $\underset{ij}{\otimes}$ will stand for the relative tensor product of the two triangle modules that contain the edge (ij). Whether it is a tensor product with respect to ε or ε^o can be unambiguously recovered from the order of the modules in the tensor product. This is because each internal edge (ij) (of a planar 2-complex) has opposite relative orientation to its two neighbour faces. For example

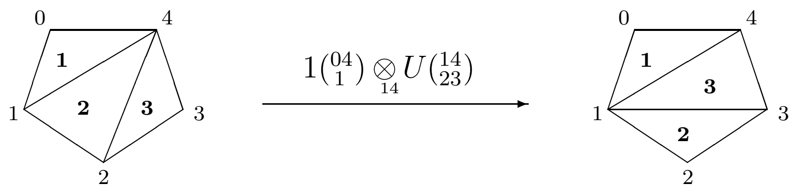
$$\binom{02}{1} \underset{02}{\otimes} \binom{04}{2} \underset{24}{\otimes} \binom{24}{3} = \mathcal{H}_{\alpha_{02}} \underset{\varepsilon}{\otimes} {}_{\alpha_{01}}\mathcal{H}_{\alpha_{12}} \underset{\varepsilon^o}{\otimes} {}_{\alpha_{02}}\mathcal{H}$$

Sometimes it is unavoidable to use brackets because $\underset{ij}{\otimes}$ refers to two triangles that are not consecutive ones in the tensor product. For example in

$$[\binom{03}{1} \underset{13}{\otimes} \binom{13}{2}] \underset{03}{\otimes} \binom{04}{3} = [\mathcal{H}_{\alpha_{12}} \underset{\varepsilon^o}{\otimes} {}_{\alpha_{02}}\mathcal{H}]_{\alpha_{03}} \underset{\varepsilon}{\otimes} {}_{\alpha_{01}}\mathcal{H}$$

These brackets therefore have nothing to do with associativity of the tensor product. They reflect rather the poor capability of our one dimensional writing to express two dimensional facts.

Now we are ready to formulate the pentagon equation. In our notation the equation of Definition 5.6.ii of [5] takes the form as Figure 1. The boldface numbers in the argument of the flip map refer to factors of the tensor product that forms the domain of Σ. E.g. $\Sigma(\mathbf{12},\mathbf{3})$ maps $\xi \otimes \eta \otimes \zeta$ to $\zeta \otimes \xi \otimes \eta$. Up to the flip maps, which serve only for permuting the tensor product factors in linear writing, the above commutative diagram is a pentagon rather than an octogon. The reader may find it amusing to draw the eight pentagonal figures corresponding to the eight vertices of Figure 1, each of them having vertices numbered (01234), have two diagonals one for each $\underset{ij}{\otimes}$ symbol, and have triangular faces numbered according to their order in the tensor product. Let stand here one edge of Figure 1 for example:



After acquainting the equation we have to show that it is a consequence of the V-pentagon (2.1). At first we identify the eight corners in Figure 1 with subspaces of $\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}$. With the notation $E = VV^*$, $\hat{E} = V^*V$ we can write

$$\begin{aligned}
\binom{04}{1} \underset{14}{\otimes} \binom{14}{2} \underset{24}{\otimes} \binom{24}{3} &= \hat{E}_{12}\hat{E}_{23}(\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}) \\
\binom{04}{1} \underset{14}{\otimes} [\binom{13}{2} \underset{13}{\otimes} \binom{14}{3}] &= \hat{E}_{13}E_{23}(\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}) \\
\binom{04}{1} \underset{14}{\otimes} \binom{14}{3} \underset{13}{\otimes} \binom{13}{2} &= \hat{E}_{12}E_{32}(\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}) \\
\binom{13}{2} \underset{13}{\otimes} [\binom{04}{1} \underset{14}{\otimes} \binom{14}{3}] &= E_{13}\hat{E}_{23}(\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}) \\
\binom{13}{2} \underset{13}{\otimes} \binom{03}{1} \underset{03}{\otimes} \binom{04}{3} &= \hat{E}_{21}E_{23}(\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}) \\
[\binom{03}{1} \underset{13}{\otimes} \binom{13}{2}] \underset{03}{\otimes} \binom{04}{3} &= E_{13}\hat{E}_{12}(\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}) \\
\binom{02}{1} \underset{02}{\otimes} \binom{04}{2} \underset{24}{\otimes} \binom{24}{3} &= E_{12}\hat{E}_{23}(\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}) \\
\binom{02}{1} \underset{02}{\otimes} \binom{03}{2} \underset{03}{\otimes} \binom{04}{3} &= E_{12}E_{23}(\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H})
\end{aligned}$$

Inserting $V = VV^*V = EV\hat{E}$ into the V-pentagon (2.1) we obtain

$$E_{23}V_{23}(\hat{E}_{23}E_{12})V_{12}\hat{E}_{12} = E_{12}V_{12}(\hat{E}_{12}E_{13})V_{13}(\hat{E}_{13}E_{23})V_{23}\hat{E}_{23} \ .$$

Multiplying with projections from the left and right and inserting appropriate flip maps

$$\begin{aligned}
(E_{12}E_{23})V_{23}(\hat{E}_{23}E_{12})V_{12}(\hat{E}_{12}\hat{E}_{23}) &= (E_{12}E_{23})V_{12}(\hat{E}_{12}E_{13})\Sigma_{1,2}(\hat{E}_{21}E_{23}) \\
&\quad V_{23}(\hat{E}_{23}E_{13})\Sigma_{12,3}(\hat{E}_{12}E_{32})\Sigma_{2,3}(\hat{E}_{13}E_{23}) \\
&\quad V_{23}(\hat{E}_{12}\hat{E}_{23}) \qquad (6.52)
\end{aligned}$$

The eight different projections in the parentheses correspond precisely to the eight corners of the diagram in Fig.1. The V and Σ operators, together with

their neighbour projections, in turn produce precisely the eight maps of the diagram. In order to see this one should check correspondences like

$$\begin{aligned}
U\binom{04}{12} \underset{24}{\otimes} 1\binom{24}{3} &\equiv \hat{E}_{23}V_{12}\hat{E}_{23} = (E_{12}\hat{E}_{23})V_{12}(\hat{E}_{12}\hat{E}_{23}) \\
1\binom{02}{1} \underset{02}{\otimes} U\binom{04}{23} &\equiv E_{12}V_{23}E_{12} = (E_{12}E_{23})V_{23}(E_{12}\hat{E}_{23}) \\
1\binom{04}{1} \underset{14}{\otimes} \Sigma(\binom{13}{2},\binom{14}{3}) &\equiv (\hat{E}_{12}E_{32})\Sigma_{23}(E_{23}\hat{E}_{13})
\end{aligned}$$

and five other ones. This finishes the proof of that every unital MPI V determines a pseudo-multiplicative unitary U by restriction to range and domain. As a byproduct we obtained a geometric interpretation of the equations in terms of trimodules, or 2-simplex modules, $\mathcal{H}$ over A^L in which U plays the role of Ocneanu's 3-cocycle.

Now we turn to the opposite construction when we are given a pseudo-multiplicative unitary U and want to construct a multiplicative partial isometry V that reproduces U by restriction. This task will be a simple one mainly because we can prove only that the resulting V is an MPI and we leave it open whether V is unital.

Let N be a finite dimensional C^*-algebra with a faithful positive linear functional $\nu: N \to \mathbb{C}$ of index 1. Let $\beta, \alpha, \hat{\beta}$ be actions of N, N^o, and N, respectively on a finite dimensional Hilbert space $\mathcal{H}$ that commute with each other. Finally, let $U: \mathcal{H}_{\hat{\beta}} \underset{\nu^o}{\otimes} {}_{\alpha}\mathcal{H} \to \mathcal{H}_{\alpha} \underset{\nu}{\otimes} {}_{\beta}\mathcal{H}$ be a pseudo-multiplicative isometry.

Since the relative tensor products can be identified as subspaces in $\mathcal{H}\otimes\mathcal{H}$ via the projections (5.36), we can immediately define a partial isometry

$$V := EU\hat{E}\,, \quad \text{where} \quad E = (\alpha\otimes\beta)(e_\nu),\ \hat{E} = (\hat{\beta}\otimes\alpha)(e_\nu)\,. \tag{6.53}$$

Then $VV^* = E$ and $V^*V = \hat{E}$. Defining the algebras $A^L := \beta(N)$, $A^R := \alpha(N)$, and $\hat{A}^R := \hat{\beta}(N)$ the four intertwiner relations for U become the intertwiner relations (6.47a-d). These in turn are equivalent to the equations (2.3), (2.2), (2.4), and (2.9), respectively. The pentagon equation (2.1) can now be obtained by arguing backwards with equation (6.52). This proves that V is a multiplicative isometry.

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